Flat rotational curves without dark matter

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Abstract

We consider the singular configurations of gravitating gas [1] which could be used as a model for disk galaxies. The simplest steady configuration, which corresponds to rotation of stars around center gives flat rotational curve, provided the density distrubution has a tail, decaying as $1/r$. This result based exclusively on Newtonian gravity and does not involve dark matter.

1 Introduction

In the present paper we continue started in [1] study of the dynamics of the system called gravitating gas. This system is a kind of continuous media which is constituted by the particles (stars) interacting with each other only gravitationaly i.e. according Newton's law. As a mechanical system it could be considered as a continuous limit of the N-particles system described by Hamiltonian:

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m_i} - \sum_{i\neq j}^{N} \frac{Gm_i m_j}{|\vec{x}_i - \vec{x}_j|}, \qquad (1)$$

where $\vec{x}_i, \vec{p}_i$ are canonical coordinates of particles (stars) with masses m_i, G-gravitational constant.

Speaking about galaxy, where $N \sim 10^{13}$ it seems reasonable to consider for the system (1) the limit $N \to \infty$ (with assumption $m_i = m$) and substitute the configuration of particles by continuous distribution of its density and velocity, as it is done in the theory of gas or fluid.

Certainly, we have to have in mind that the resulting system– gravitating gas is a very strange gas due to attractive interaction of its constituents. In the usual gas or fluid, the constituents (atoms or moleculars) have a repulsive interaction and being put in a volume it spread in space, filling after

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some time the whole volume with uniform density. As it was proved in [1], the gravitating gas, because of attractive interaction may form isolated steady configuration with asymptotically vanishing density. In particularly, the galaxy named Hoag's object may be an example of such gravitational soliton [1].

Here we are going to consider another kind of steady configurations of gravitating gas which has a singular distribution of density

$$\rho(\vec{x}) = \delta(z)\rho(r, \phi), \tag{2}$$

where z, r, ϕ are cylindrical coordinates. This kind of solutions may be considered as an idealized model of disk galaxies.

Apart from the application on the galactic scale these singular solutions may be used for the explanation of planetary rings such as Saturn ring with its surprisingly thin structure formed exclusively by gravity.

2 Equations of motion

In fluid (gas) mechanics there are two different pictures of description. The first, usually refereed as Eulerian, uses as the coordinates the space dependent fields of velocity and density. The second, Lagrangian description, uses the coordinates of the particles $\vec{x}(\xi_i, t)$ labeled by the set of the parameters ξ_i (the numbers of particles in (1)), which could be considered as the initial positions $\vec{\xi} = \vec{x}(\xi_i, t = 0)$ and time t. The useful physical assumption is that the functions $\vec{x}(\xi_i, t)$ define a diffeomorphism of $D \subseteq R^3$ and the inverse functions $\vec{\xi}(x_i, t)$ should also exist.

$$\begin{aligned} x_j(\xi_i, t)\Big|_{\vec{\xi}=\vec{\xi}(x_i,t)} &= x_j, \\ \xi_j(x_i, t)\Big|_{\vec{x}=\vec{x}(\xi_i,t)} &= \xi_j. \end{aligned} \tag{3}$$

The density of the particles in space at time t is

$$\rho(\vec{x}, t) = \int d^3\xi \rho_0(\xi_i)\delta(\vec{x} - \vec{x}(\xi_i, t)), \tag{4}$$

where $\rho_0(\xi)$ is the initial density at time $t = 0$. The velocity field $\vec{v}$ as a function of coordinates $\vec{x}$ and t is:

$$\vec{v}(x_i, t) = \dot{\vec{x}}(\vec{\xi}(x_i, t), t), \tag{5}$$

where $\vec{\xi}(x,t)$ is the inverse function (3). The velocity also could be written in the following form:

$$\vec{v}(x_i,t) = \frac{\int d^3\xi \rho_0(\xi_i)\dot{\vec{x}}(\xi_i,t)\delta(\vec{x}-\vec{x}(\xi_i,t))}{\int d^3\xi \rho_0(\xi_i)\delta(\vec{x}-\vec{x}(\xi_i,t))}, \tag{6}$$

or

$$\rho(x_i,t)\vec{v}(x_i,t) = \int d^3\xi \rho_0(\xi_i)\dot{\vec{x}}(\xi_i,t)\delta(\vec{x}-\vec{x}(\xi_i,t)). \tag{7}$$

Let us calculate the time derivative of the density using its definition (4) :

$$\begin{aligned}\dot{\rho}(x_i,t) &= \int d^3\xi \rho_0(\xi_i)\frac{\partial}{\partial t}\delta(\vec{x}-\vec{x}(\xi_i,t)) \\ &= \int d^3\xi \rho_0(\xi_i)\left(-\dot{\vec{x}}(\xi_i,t)\right)\frac{\partial}{\partial \vec{x}}\delta(\vec{x}-\vec{x}(\xi_i,t)) \\ &= -\tfrac{\partial}{\partial \vec{x}}\textstyle\int d^3\xi \rho_0(\xi_i)\dot{\vec{x}}(\xi_i,t)\delta(\vec{x}-\vec{x}(\xi_i,t)) \\ &= -\frac{\partial}{\partial \vec{x}}\rho(x_i,t)\vec{v}(x_i,t)\end{aligned} \tag{8}$$

In such a way we verify the continuity equation of fluid dynamics:

$$\dot{\rho}(x_i,t) + \vec{\partial}\Big(\rho(x_i,t)\vec{v}(x_i,t)\Big) = 0. \tag{9}$$

.

Using the coordinates $\vec{x}(\xi_i,t)$ as a configurational space variables we can write the Lagrangian for the continuous generalization of the system, described by (1)

$$L = \int d^3\xi \rho_0(\xi_i)\frac{m\dot{\vec{x}}^2(\xi_i,t)}{2} + \frac{Gm^2}{2}\int d^3\xi d^3\xi' \frac{\rho_0(\xi_i)\rho_0(\xi_i')}{|\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)|}, \tag{10}$$

The equations of motion, which follow from the Lagrangian (10) have the form:

$$m\ddot{\vec{x}}(\xi_i,t) + Gm^2\int d^3\xi' \rho_0(\xi_i')\frac{\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)}{|\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)|^3} = 0. \tag{11}$$

Now we need to translate the equations (11) on to the language of Euler variables. For this let us differentiate both sides of the equation(6) with respect to time:

$$\begin{aligned}\frac{\partial}{\partial t}\rho(x_i,t)\vec{v}(x_i,t) &= \int d^3\xi \rho_0(\xi_i)\ddot{\vec{x}}(\xi_i,t)\delta(\vec{x}-\vec{x}(\xi_i,t)) \\ &+ \textstyle\int d^3\xi \rho_0(\xi_i)\dot{\vec{x}}(\xi_i,t)\frac{\partial}{\partial t}\delta(\vec{x}-\vec{x}(\xi_i,t)).\end{aligned} \tag{12}$$

Substituting $\ddot{\vec{x}}(\xi_i, t)$ from the equation (11) and transforming the second term, as we did in derivation of the equation (7)we obtain:

$$\frac{\partial}{\partial t}\rho(x_i,t)\vec{v}(x_i,t) = -\frac{\partial}{\partial x_k}\Big(\rho(x_i,t)\vec{v}(x_i,t)v_k(x_i,t)\Big) + \int d^3\xi\rho_0(\xi_i)\Big[-Gm\int d^3\xi'\rho_0(\xi_i')\frac{\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)}{|\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)|^3}\Big]\delta(\vec{x}-\vec{x}(\xi_i,t)) \tag{13}$$

To transform the last integral we first perform integration over the ξ:

$$\int d^3\xi\rho_0(\xi_i)\Big[-Gm\int d^3\xi'\rho_0(\xi_i')\frac{\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)}{|\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)|^3}\Big]\delta(\vec{x}-\vec{x}(\xi_i,t)) = -Gm\rho(x_i,t)\int d^3\xi'\rho_0(\xi_i')\frac{\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)}{|\vec{x}-\vec{x}(\xi_i',t)|^3}. \tag{14}$$

Now let us insert in the integral over ξ' the unity

$$1 = \int d^3y\delta(\vec{y}-\vec{x}(\xi_i',t)) \tag{15}$$

and change the order of integration. Finally we arrive at

$$\int d^3\xi\rho_0(\xi_i)\Big[-Gm\int d^3\xi'\rho_0(\xi_i')\frac{\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)}{|\vec{x}(\xi_i,t)-\vec{x}(\xi_i',t)|^3}\Big]\delta(\vec{x}-\vec{x}(\xi_i',t) = = Gm\rho(x_i,t)\frac{\partial}{\partial\vec{x}}\int d^3y\frac{\rho(y_i,t)}{|\vec{x}-\vec{y}|}. \tag{16}$$

The equation of motion (11) in terms of Euler variables takes the following form:

$$\frac{\partial}{\partial t}\rho(x_i,t)\vec{v}(x_i,t) + \frac{\partial}{\partial x_k}\Big(\rho(x_i,t)\vec{v}(x_i,t)v_k(x_i,t)\Big) = = Gm\rho(x_i,t)\frac{\partial}{\partial\vec{x}}\int d^3y\frac{\rho(y_i,t)}{|\vec{x}-\vec{y}|}. \tag{17}$$

Using the continuity equation we can rewrite this equation in more simple form:

$$\frac{\partial}{\partial t}\vec{v}(x_i,t) + v_k(x_i,t)\frac{\partial}{\partial x_k}\vec{v}(x_i,t) = Gm\frac{\partial}{\partial\vec{x}}\int d^3y\frac{\rho(y_i,t)}{|\vec{x}-\vec{y}|} \tag{18}$$

But this is possible only if density does not vanish in some domain. If for example we want to study configurations with density like given by (2), we must keep the equation of motion in the form (17).

Indeed, let us consider the configuration of our system with velocity and density of the type

$$\begin{aligned} \rho(\vec{x},t) &= \delta(z)\rho(x,y,t) \\ v_i &= (v_1(x,y,t), v_2(x,y,t), 0). \end{aligned} \tag{19}$$

Substituting (19) into (17) we obtain:

$$\begin{aligned} &\frac{\partial}{\partial t}\delta(z)\rho(x,y,t)(v_1(x,y,t),v_2(x,y,t),0) + \\ &\frac{\partial}{\partial x_a}\Big(\delta(z)\rho(x,y,t)v_a(x,y,t)(v_1(x,y,t),v_2(x,y,t),0)\Big) = \\ &G\ m\ \delta(z)\rho(x,y,t)(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\int dx'dy'\frac{\rho(x',y',t)}{\sqrt{(x-x')^2+(y-y')^2+z^2}}, \end{aligned} \tag{20}$$

where index a takes the values $1,2$. Apparently, the derivative of integral with in the r.h.s of (20) with respect to z vanishes after multiplication on $\delta(z)$, because the integral is even function of z, so the third component in both sides of (20) disappears and after making use of the continuity equation we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}v_a(x,y,t) + v_b(x,y,t)\frac{\partial}{\partial x_b}v_a(x,y,t) = \\ &Gm\frac{\partial}{\partial x_a}\int dx'dy'\frac{\rho(x',y',t)}{\sqrt{(x-x')^2+(y-y')^2}}, \end{aligned} \tag{21}$$

where indexes a,b take the values $1,2$. This equation has to be completed with continuity equation, which now has the following form:

$$\frac{\partial}{\partial t}\rho(x,y,t) + \frac{\partial}{\partial x_a}\Big(\rho(x,y,t)v_a(x,y,t)\Big) = 0 \tag{22}$$

Note, that as it should be anticipated, the equations (21) and (22) took the usual form of 2-dimensional gas dynamics, but the potential inherited its 3-dimensional form.

3 Simplest steady solution and its rotational curves.

Here we are going to consider the time independent configurations of gravitating gas, therefore all time derivatives in the equations (21) and (22) will disappear.

$$v_b(x,y)\frac{\partial}{\partial x_b}v_a(x,y) = Gm\frac{\partial}{\partial x_a}\int dx'dy'\frac{\rho(x',y')}{\sqrt{(x-x')^2+(y-y')^2}},$$
$$\frac{\partial}{\partial x_a}\Big(\rho(x,y,t)v_a(x,y,t)\Big) = 0. \tag{23}$$

One of the simplest configuration of gravitating gas describes the rotation of the particles (stars) around central point. For this configuration we have:

$$v_a = v(r)(\frac{y}{r},-\frac{x}{r}),$$
$$\rho = \rho(r), \qquad r = \sqrt{x^2+y^2}. \tag{24}$$

This ansatz satisfies the continuity equation and the main equation give a relation between velocity $v(r)$ and density $\rho(r)$:

$$v^2(r) = -GMr\partial_r\int dr'r'\rho(r')t(r,r'), \tag{25}$$

where the kernel $t(r,r')$ is given by:

$$t(r,r') = \int_0^{2\pi}d\phi\frac{1}{\sqrt{r^2+r'^2-2rr'cos\phi}} = \frac{4}{r+r'}\mathbf{K}(k), \tag{26}$$
$$k^2 = \frac{4rr'}{(r+r')^2}$$

and $\mathbf{K}(k)$ the complete elliptic integral of the 1-st kind. So, from this solution follows that any distribution of density is possible and and we have to consider the stability of solution in order choose $\rho(r)$.

The equation (25) shows that is in case of flat disk configuration of stars the famous Newton's theorem is not valid. Of course, this theorem which states that [2]

$$v^2(r) = Gm\frac{M(r)}{r} \tag{27}$$

where $M(r)$ is the total mass inside the orbit, is true only for circular orbits in spherically symmetric density distribution. In our case we can rewrite (25) in the following form using modular transformation of modulus k:

$$v^2(r) = -Gmr\partial_r\Big(\int_0^r dr'r'\rho(r')\frac{4}{r}\mathbf{K}(\frac{r'}{r}) + \int_r^\infty dr'r'\rho(r')\frac{4}{r'}\mathbf{K}(\frac{r}{r'})\Big). \qquad (28)$$

The first term in the r.h.s. has the factor $\frac{1}{r}$, but dependence of r is present also in the integrands of both terms (the dependence from the integration limits is cancelled of course).

So, what we can say about typical rotational curves i.e. the functions $v(r)$ for different densities? Apparently, if the density is strictly localized within the circle with radius R, when, for $r \gg R$ we obtain the usual result:

$$v^2(r) = Gm\frac{M}{r}, \qquad (29)$$

where M is the total mass (which all is in the circle $r = R$).

In the disk galaxies almost all mass is concentrated near the center decaying exponentially $e^{-r/h}$ with $h \sim 1 - 5 \quad kpc$ [2]. In this case (29) could be applied for $r \gg h$. But, in the same time the disk of the galaxy, as it seen is much bigger then h, therefore we may suggest that there is a tail in the density distribution which drops slowly, e.g. as a power of r.

In order to explore the influence of such kind of densities, we need to know properties of the kernel $t(r, r')$ in the integral(25). Apparently it is a symmetric function of r, r' and has the logarithmic singularity when $r \to r'$ (the singularity of elliptic integral $\mathbf{K}(k)$ for $k \to 1$). Unfortunately there is only one case when we can express the integral (25) in terms of elementary functions: for density

$$\rho(r) = \frac{c}{r} \qquad (30)$$

the integral is given by

$$\int dr'r'\rho(r')t(r,r') = -c\ 2\pi \ln\ r + C', \qquad (31)$$

where C' is a divergent constant, which is inessential for calculation of $v(r)$:

$$v^2(r) = G\ m\ c\ 2\pi \qquad (32)$$

So, the rotational curve for $\rho(r)$, produced by the density (30) is flat! Certainly, in order to have flat rotational curve, it is sufficient to have the behaviour of density given by (30) only asymptotically, starting e.g. from the distances $\sim 6-8kpc$. For smaller distance the exponential distribution prevails, not changing the behavior at large distance.

As it follows from the analysis of the integral in (25), the density (30) is the limiting case for the convergence of integral which defines $v^2(r)$. If asymptotically

$$\rho(r) = \frac{1}{r^{1+\alpha}}, \qquad \alpha > 0 \tag{33}$$

then we have decreasing rotational curve.

4 Discussion.

The result we obtained in the previous section is not very unexpected and is the direct consequence of the flat density distribution. In order to explain it let us derive it in a more traditional way without referring to gravitational gas, considering the motion of particle in axially symmetric density distribution $\rho(x_i) = \rho(r, \cos\theta)$, where r, θ, ϕ now are spherical coordinates. Then the equation of motion is given by

$$\ddot{\vec{x}}(t) = Gm\vec{\partial} \int dr' r'^2 d\phi' d\cos\theta' \frac{\rho(r', \cos\theta')}{\sqrt{r^2 + r'^2 - 2rr'\cos\Theta}}, \tag{34}$$

where Θ is the angle between $\vec{x}$ and $\vec{x}'$:

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{35}$$

Now let us assume that the test particle moves along circular orbit with angular velocity $\omega(r)$, therefore

$$\ddot{\vec{x}}(t) = -\omega^2(r)\vec{x}(t) = -\frac{v^2(r)}{r^2}\vec{x}(t), \tag{36}$$

from where we obtain:

$$v^2(r) = Gmr\partial_r \int dr' r'^2 d\phi' d\cos\theta' \frac{\rho(r', \cos\theta')}{\sqrt{r^2 + r'^2 - 2rr'\cos\Theta}}, \tag{37}$$

Using well-known decomposition of the square root in (34) we can expand the r.h.s. of (37) in the following form:

$$\begin{aligned} v^2(r) = Gmr\partial_r \sum_{n=0}^{\infty} \Big[&\int_0^r dr' r'^2 d\phi' d\cos\theta' \rho(r', \cos\theta') \frac{r'^n}{r^{n+1}} P_n(\cos\Theta) \\ &+ \int_r^{\infty} dr' r'^2 d\phi' d\cos\theta' \rho(r', \cos\theta') \frac{r^n}{r'^{n+1}} P_n(\cos\Theta) \Big]. \end{aligned} \tag{38}$$

If the density is spherically symmetric, the integration over angles θ' and ϕ' leaves only the first term in the sum and we immediately recover Newton's theorem. In any general case, all terms of the sum are non-zero and we have complicated behaviour of rotational curve. If for example the density has the form we had considered above, i.e. $\rho \sim \rho(r)\delta(rcos(\theta))$, and the orbit lies in the plane xy we are coming to the equation from the previous section. Certainly, this derivation can not substitute the whole dynamics of gravitating gas we had considered, because the potential we used there is created by the same particles, which move in the potential.

Now we can estimate the order of density we need to fit the observation data of the velocities on the flat part of the rotational curves. For typical disk galaxy these velocities reach the $150 - 300 km./sec$. If we shall take as mass of the particle to be equal to the mass of our Sun —$2 \cdot 10^{30}$ kg, the coefficient $GM2\pi$ will be $\sim 8,4 \cdot 10^{20} m^3/sec^2$. From here we obtain for the constant c the values $(0,27 - 1,1) \cdot 10^{-10} m^{-1}$, which corresponds to $(0,8 - 3,3) \cdot 10^6 pc^{-1}$. The surface density therefore becomes

$$\rho(r) = \frac{1}{r} \times (0,8 - 3,3) \cdot 10^6 pc^{-2}, \tag{39}$$

where r is measured in pc. On the distance of e.g. 10 kpc, this formula gives us the surface density $\sim (0,8 - 3,3) \cdot 10^2$ stars on square parsec. These numbers is not in contradiction with observational results [7].

The precise measurement of velocities of stars in various galaxies which was done during last 30 years [3],[4] made it possible to discover the peculiar behaviour of the rotational curves. This in turn led one part of community of astronomers and astrophysicists to the hypothesis of dark matter, the other to modification of gravitational law [5],[6]. Later another confirmation of the existance of dark matter has emerged: the data on gravitational lensing and cosmic microwave background radiation. However, the argument, given by the rotational curves stays aside as a completely model independent measurement, not based on any assumptions of global structure of Universe. That is the reason why it is so important to explore all possibility within traditional physics and I hope that the present paper is one more step in this direction.

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References

[1] G.Pronko, "Solitons in gravitating gas. Hoag's object", Theoretical and Mathematical Physics, vol.146 (1), p. 85-94, (2006), hep-th/0503167,

[2] J.Gallagher, L.Sparke, L.Matthews, "Disk Galaxies", Encyclopedia of Astronomy & Astrophysics, IOP Publishing, 2006.

[3] V. Rubin, W.K. Ford, Jr, "Rotation of the Andromeda Nebula from a spectroscopic survey of emission region", Astrophysical Journal, v. 159, p. 379,(1970)

[4] V. Rubin, W.K. Ford, Jr, N.Thonnard, "Rotational properties of 21 Sc galaxies with large range of luminosities and radii from NGC 4605 (R=4 kpc) to UGC 2885 (R=122 kpc)", Astrophysical Journal, v.238, p. 471, (1980)

[5] J.D. Bekenstein, "Relativistic gravitational theory for the modified Newtonian dynamics paradigm", Phys. Rev. D, v.70, 088509,(2004)

[6] J.W. Moffat, "Scalar tensor vector gravity theory", Jounnal of Cosmology and Astroparticle Physics, v. 3, p.4, (2006)

[7] T.Takamiya, Y. Sofue, "Radial distribution of the Mass-to-Luminosity ratio in spiral galaxies and massive dark cores", Astrophysical Journal, ApJ. 534,670, (1999)